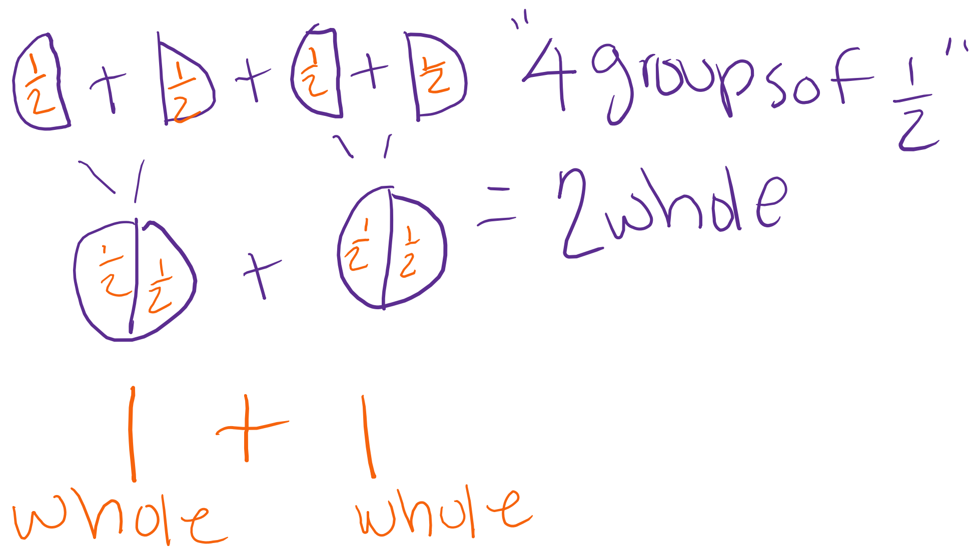
<u>Teaching Multiplication and Division of Fractions and Decimals</u>

It is more efficient to teach these operations on fractions first, so that students can call on their understanding of those concepts to make sense of decimals.

Multiplication of Fractions
Beginning with multiplication of fractions:
Connection: Multiplication means "Groups of"

So 4 X ½ means four groups of ½. Have students draw this and they will realize it equals 2.



Student should understand that multiplication is "commutative". That means that if we multiply two factors we can multiply them in any order and get the same product. (if kids don't know this, review it: 3 x 4 = 12 and 4 x 3 = 12. Draw pictures of 3 groups of 4 and four groups of 3.

So the expression 4 x ½ from above can also be ½ x 4
In this case, we say "One half of a group of four" or, more simply, "one half of four".
Most kids know that half of four is two (same answer as above).
Big Connection: Multiplication means groups of.
When reading mathematical expressions with fractions, decimals, and percents, we often take the "X" multiplication symbol to mean "of".

One third of 12 is three.
1/3 x 12 = 3

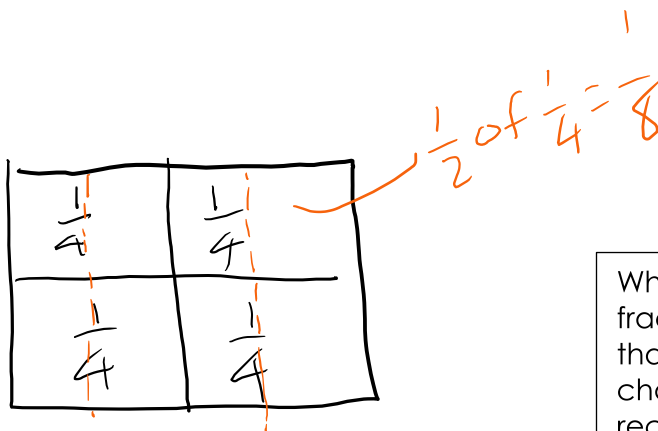
Now, draw students' attention to the fact that when we multiply by fractions (smaller than one, such as ½ or 1/3 like our examples above), our product is *smaller*. This is important.
Above, when we multiplied 4 by ½ our answer is *smaller* than 4.
Up until this point, students have seen whole number multiplication, so they are used to multiplication making numbers bigger.
4 x 5 = 20, 6 x 8 = 48 and so on.

Activity

There are some fun ways to do this, like using a "commit and toss", or "vote with your feet" activity.
Pose the statement: "When I multiply two numbers, my product is always bigger or equal to my factors". Up until now, this has been true for kids. Some might conjecture about multiplication by zero or one. You can do this activity before the lesson, and redo it the next day to see if anyone changes their mind.

Ask students, "What is half of one quarter". Some will guess one eighth.

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$
Have them draw it.



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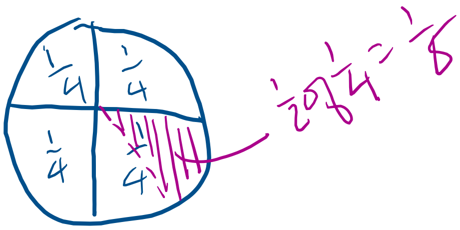


When representing fractions, help kids note that the "whole" can change. Here I have a rectangular whole and also a circular model. We also need to point out we sometimes draw fractions of a whole, and sometimes fractions of a set.

Kids should know that 1/8 is smaller than ½ and smaller than ¼
So big idea: *If I multiply fractions, I actually make the product smaller.*
You may have to qualify that this is true for fractions < 1 (fractions smaller than one whole) because $\frac{10}{3}$ is a fraction, but multiplying by this will make your product greater, of course.

Here we can teach multiplying fractions symbolically

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$

Soon students will skip this step.

Practice mental math strategies in a number talk:

what is $\frac{1}{3}$ of 12? $\frac{1}{3} \times 12 =$

$\frac{1}{4}$ of 20? $\frac{1}{4} \times 20 =$

$\frac{1}{2}$ of 10? $\frac{1}{2} \times 10 =$

Kids will know answers by logic.

** BIG CONNECTION: Did they notice that taking a third of something is just dividing by 3? Did they notice taking $\frac{1}{4}$ of something is just dividing 4?

** BIG CONNECTION!! Fractions are another way of writing division!!

$$\frac{1}{2} \times 10 = \frac{1}{2} \times \frac{10}{1} = \frac{10}{2} = 10 \div 2 = 5$$

caution

If I were writing this on the board I would put one step below the other. Avoid letting kids get in the habit of writing a bunch of = signs in a row.

$\frac{1}{2} \times 10 = \frac{1}{2} \times \frac{10}{1} = \frac{10}{2} = 10 \div 2 = 5$

"one half of ten is 5"

Fractions really are another way of writing division

You can tell students that as they progress through to high school, they'll see this $\div$ sign less and less! For $10 \div 2$ we will write $\frac{10}{2}$. For $x \div 4$ we will write $\frac{x}{4}$.

Now, go back to this:

what is $\frac{1}{3}$ of 12?	$\frac{1}{3} \times 12 =$	So what is $\frac{2}{3}$ x 12?
$\frac{1}{4}$ of 20?	$\frac{1}{4} \times 20 =$	$\frac{3}{4}$ x 20
$\frac{1}{2}$ of 10?	$\frac{1}{2} \times 10 =$	

Well, if $\frac{1}{3}$ of 12 is 4, then $\frac{2}{3}$ would be 2 of those, so 8.

If one quarter of 20 is 5, then three quarters is three of those, so 3 fives is 15. $\frac{3}{4} \times 20 = 15$

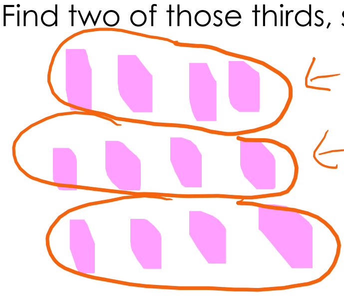
Look how easy it is to multiply by fractions!
Now lets use the symbolic algorithm

$\frac{2}{3} \times 12$ $\qquad$ $\frac{2}{3} \times \frac{12}{1} = \frac{24}{3}$

$\frac{24}{3} = 8$ $\qquad$ $\boxed{\frac{2}{3} \times 12 = 8}$

Here we are using fractions of a set

Be sure to give kids a representation to make sense of this. Draw a set of 12, circle thirds. Find two of those thirds, so that's eight.



← one third
← one third
Two thirds

"$\frac{2}{3}$ of 12 is 8"
"$\frac{2}{3} \times 12 = 8$"

Multiplying 12 by 2/3 mentally is dividing 12 by 3, then doubling it.
When we do the algorithm, we are doubling first, then dividing by three. Same thing!

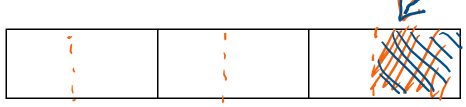
$\frac{2}{3} \times 12$ $\qquad$ $\frac{2}{3} \times \frac{12}{1} = \frac{24}{3}$

doubling 12

$\frac{24}{3} = 8$ $\qquad$ $\boxed{\frac{2}{3} \times 12 = 8}$

Then dividing by 3.

$\frac{1}{2} \times \frac{1}{3}$ (half of a third)



Half of a third is one sixth of the whole

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

Here we are using fractions of a whole

Once we've had some practice, lets make sure we get kids to "cross cancel". Cross out common factors before we multiply. ***This is not cross multiplying—some students will want to call it that. It is NOT.*

$$\frac{2}{3}\times 12 \qquad \frac{2}{\cancel{3}_{1}}\times\frac{\cancel{12}^{4}}{1} = \frac{8}{1} = 8$$

Another model.

$$\frac{2}{3}\times\frac{12}{1} = \frac{2\times12}{3} = \frac{2\times\overbrace{2\times2\times\cancel{3}}^{12}}{\cancel{3}} = \frac{2\times2\times2}{1} = 8$$

Kids should practice cross canceling. There are few to no examples of this in the text books:

Ex: $$\frac{\cancel{15}^{3}}{\cancel{8}_{4}}\times\frac{\cancel{24}^{6\ 3}}{\cancel{40}_{8\ 4}}\times\frac{\cancel{2}}{\cancel{3}} = \frac{3}{4}$$

We cross out any common factors between any numerator + any denominator.

Another model.

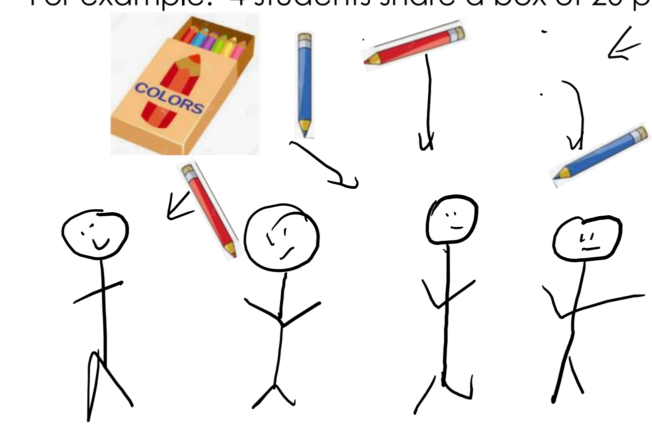
$$\frac{15}{8}\times\frac{24}{40}\times\frac{2}{3} = \frac{\overbrace{3\times\cancel{5}}^{15}\times\overbrace{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{3}}^{24}\times\cancel{2}}{\underbrace{2\times2\times\cancel{2}}_{8}\times\underbrace{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{5}}_{40}\times\cancel{3}} = \frac{3}{2\times2} = \frac{3}{4}$$

This method uses prime factors. Good to illustrate once or twice, but above method is how we would show our work.

Fraction Division

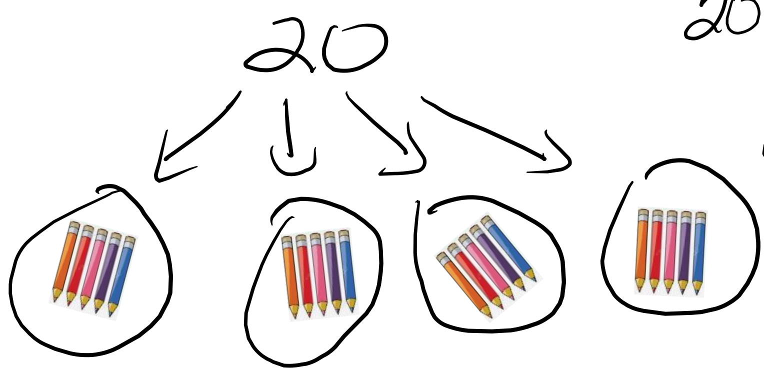
First: It is important to note that there are two kinds of division:

<u>Partitive Division:</u> Means equal sharing, or splitting a group of items into a known number of smaller groups
For example: 4 students share a box of 20 pencil crayons



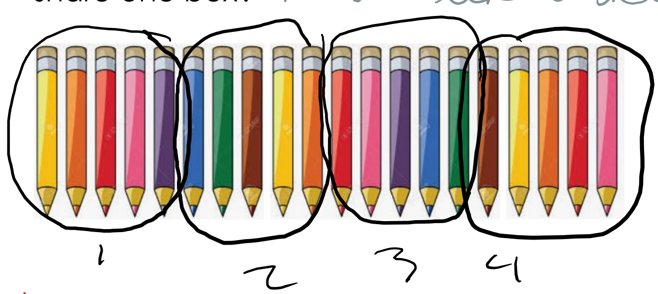
← we start dealing out all 20 crayons, and we find out each student gets 5

$20 \div 4 = 5$



← The number of groups is known.

<u>Quotative division</u> Is when we know the number of objects in each group, and we want to know how many groups we can make:
Ex: We have 20 pencil crayons in a box. Each student needs 5. How many students can share one box? This is like asking "How many groups of 5 are in 20"?

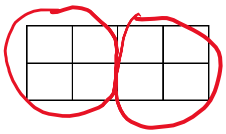


I can make 4 groups.
$20 \div 5 = 4$

* it is easiest to think of fraction division as "quotative".

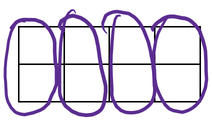
So: $8 \div \frac{1}{2} =$ is asking: "How many groups of half are in 8?"

You may want to start with $8 \div 4 =$
How many groups of 4 are in 8? Kids will know it's two. Draw it for them:



2 groups of 4.
$8 \div 4 = 2$

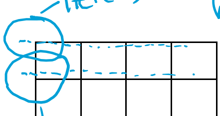
Now $8 \div 2 =$ "How many groups of 2 are in 8?"



4 groups of 2 are in 8
$8 \div 2 = 4$

So it follows: $8 \div \frac{1}{2} =$ "How many groups of one half are in 8?"

Here's two halves



Here's two halves

for each of my 8 squares there are two halves

$8 \times 2 = 16$

There are 16 halves in 8.

HEY:
$8 \div \frac{1}{2}$ is same as
8×2
(That is why when we divide by a fraction, we multiply by the reciprocal! (2 is the reciprocal of $\frac{1}{2}$.)

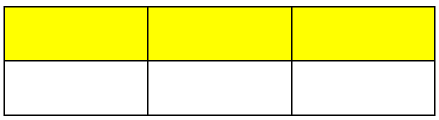
WHOA!! when I divide by a fraction my answer gets BIGGER!!

BIG IDEA!

Do not teach students to just "multiply by the reciprocal" or "invert and multiply" before explaining **why**.

Fraction divided by a fraction:

Ex: $\frac{1}{2} \div \frac{1}{6} =$ is asking "how many sixths are in one half"?



You can clearly see in this model, there are three sixths in one half. $\frac{1}{2} \div \frac{1}{6} = 3$

There are six sixths in the whole, and we are taking half of them.

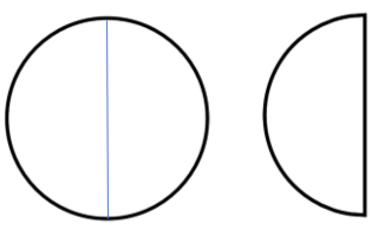
$\frac{1}{2} \times 6$

Oh. Multiply by reciprocal.

Do not teach students to just "multiply by the reciprocal" or "invert and multiply" before explaining **why**.

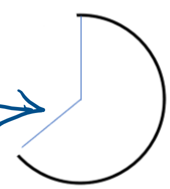
An example with a remainder: (Note that this is a more advanced model)

$1\frac{1}{2} \div \frac{2}{3} =$ This is asking "how many "two thirds" are in "one and a half"?



Here is one and a half, or three halves (review mixed and improper fractions

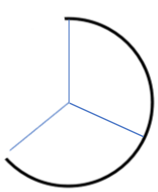
Here is two thirds

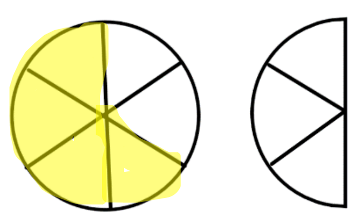


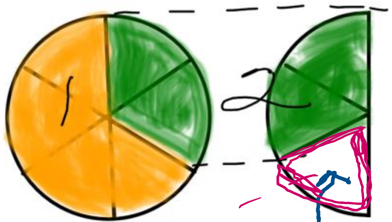
So how many of those are in $1\frac{1}{2}$?

If I partition everything into sixths, I can visualize both halves and thirds (this is why we need common denominators!)

Each $\frac{2}{3}$ piece is <u>four</u> sixths







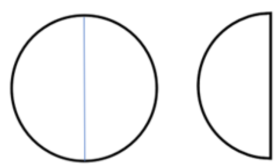
How many groups of $\frac{4}{6}$ (or $\frac{2}{3}$) are in $1\frac{1}{2}$?

Here we have 2, with 1 piece left over.

This one piece is $\frac{1}{4}$ of the "two-thirds" piece

This little leftover piece is $\frac{1}{4}$ of the yellow piece

So our quotient is $2\frac{1}{4}$.

There are two and one quarter of these  in this 

Now symbolically: $1\frac{1}{2} \div \frac{2}{3} =$

$$\frac{3}{2} \div \frac{2}{3} = \frac{3}{2} \times \frac{3}{2}$$

$$= \frac{9}{4}$$

$$= 2\frac{1}{4}$$

There are a lot of fundamental **connections** to make here!

Again, kids are used to division making a number *smaller* (we would say the quotient is smaller than the dividend)

$\underbrace{40}_{\text{Dividend}} \div 8 = 5$ and $20 \div 10 = 2$

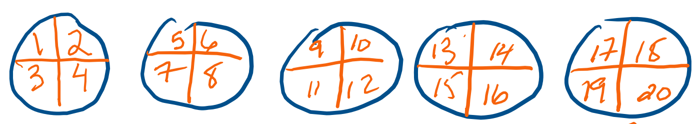
Quotient is smaller

smaller

But suddenly, when we divide by fractions (less than 1) our quotient gets *bigger* !

$5 \div \frac{1}{4} =$ means how many quarters are in 5? There are 20! 5 whole

$5 \div \frac{1}{4} = 20$ Draw it.



Each of my 5 wholes has four quarters

so $5 \div \frac{1}{4}$ becomes $5 \times 4 = 20$

Twenty quarters or twenty fourths

multiplying by the reciprocal

Caution: When drawing this diagram be sure to say "I am drawing five wholes partitioned into fourths" or "separated into fourths". Do not say "divided into fourths" because that is confusing. *Divided into* is not the same as *divided by*. Because we are teaching *division*, our words for creating regions should be *separated, partitioned, split*, but not *divided*.

Do NOT teach "multiply by the reciprocal" until you show kids ***why***.

Again, before teaching fraction division you could have kids respond to this prompt, "When I divide a number by another number, my number gets smaller – True or False?" Because up until now, that's been true. Good debate!

Now students have learned that if we multiply by a fraction less than one, our answer gets *smaller*. This is the first time they have seen that. We can use observation of patterns to verify

$8 \times 10 =$ 40

$8 \times 5 =$ 40

$8 \times 2 =$ 16

$8 \times 1 =$ 8

$8 \times \frac{1}{2} =$ 4

$8 \times \frac{1}{4} =$ 2

$8 \times \frac{1}{10} =$ $\frac{8}{10}$ or $\frac{4}{5}$ or 0.8

$8 \times \frac{1}{100} =$ $\frac{8}{100}$ or $\frac{2}{25}$ or 0.08

In our past experience, multiplication makes numbers bigger.

when our factor = 1, our number doesn't change

when our factor is less than one, our number gets smaller (product is smaller than original factor).

Leave these blanks on the board + have kids fill them in using reasoning/mental math

We want to encourage formal math language, like above, but in this case, we need a simple, clear expression of what is happening, so that kids can hold on to that reasoning and apply it to decimals. We can say **"When I multiply by a tiny number I make my answer smaller"**

* IF we multiply a number by a Factor LESS THAN 1 our product is smaller than our original factor. This becomes important as we multiply decimals.

$8 \times 0.1 = 0.8$

$8. \times 0.01 = 0.08$

$12 \times 0.01 = 0.12$

$43. \times 0.1 = 4.3$

I'm showing an "unseen" decimal at the end of whole numbers: 12 is 12.0

Be sure kids understand: "Do we agree 0.8 is smaller than 8?" ..."that 4.3 is smaller than 43?"

When I multiply by a decimal less than 1, our answer is SMALLER ∴ decimal moves LEFT.

If we divide by a fraction less than one, our answer gets *bigger*. This is the first time they have seen that. We can use observation of patterns to verify

$8 \div 8 = \underline{1}$

$8 \div 4 = \underline{2}$

$8 \div 2 = \underline{4}$

$8 \div 1 = \underline{8}$

$8 \div \frac{1}{2} = \underline{16}$ *

$8 \div \frac{1}{4} = \underline{32}$ *

$8 \div \frac{1}{10} = \underline{80}$ *

$8 \div \frac{1}{100} = \underline{800}$

Let kids fill in these blanks using reasoning.

Pattern: as the divisor gets smaller the quotient gets larger.

when the divisor equals 1, the dividend equals quotient

when divisor is Less than one, our number gets BIGGER.

← This can be used to prove why we can't divide by 0, but that's another conversation!

Plain language: " **If I divide by a big number, my answer gets smaller, but if I divide by a tiny number, my answer gets bigger**". Do we agree 800 is bigger than 8?

* This concept will become important as we begin to divide with decimals.

Remember to remind students that a decimal is another way of expressing a fraction.

$4. \div 0.1 = 40$

$18. \div 0.1 = 180$

$4.32 \div 0.1 = 43.2$

$8.91 \div 0.01 = 891$

when I divide by a decimal smaller than 1, my answer gets BIGGER

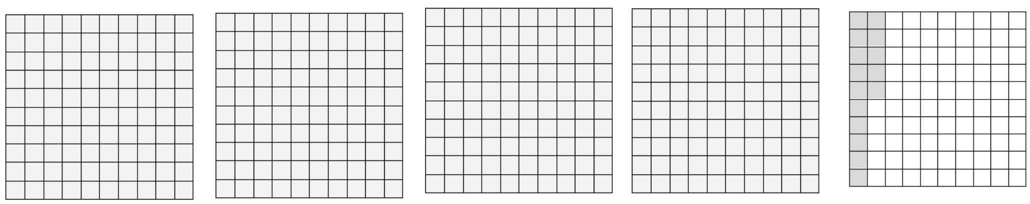
∴ decimal place moves RIGHT

$4 \div 0.1$ is $4 \div \frac{1}{10}$. How many tenths are in 4?

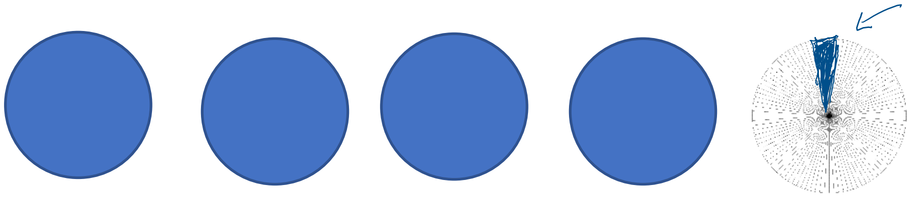
40!

Multiplication of Decimals

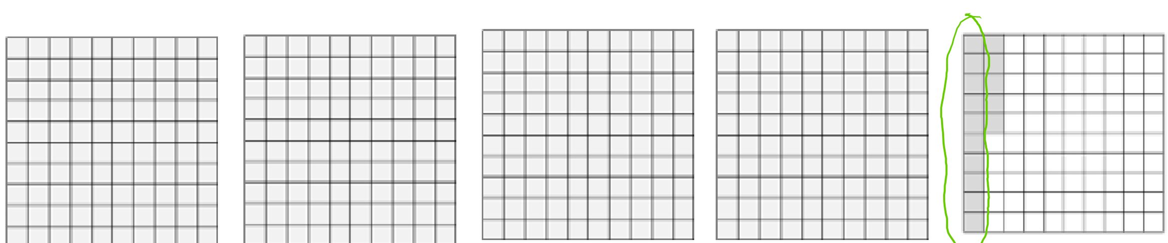
1. Review everything kids have been taught about decimals. Decimals are another way to write fractions with denominators of 10, 100, 1000 and so on. We think they know this, and they will say 4.15 as "Four and one tenth and 5 hundredths" or "four and 15 hundredths", but they might not necessarily be able to visualize it. Be sure to draw it!



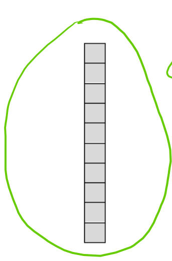
As with fractions, it is important to let kids know our "whole" can change



This circle is partitioned into hundredths and fifteen hundredths are shaded



In this model, we can point out why "four and fiftee hundredths" is the same as "four and one tenth and five hundredths". Remind them to view a column of ten as one tenth of the whole, like a ten rod.

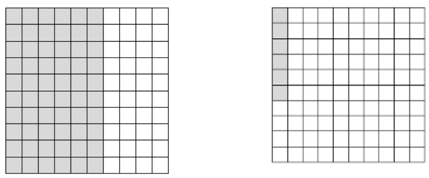


one tenth

2. Be sure students can order decimals, and understand which are larger.

0.6 is larger than 0.06

Kids get confused about decimals like 2.289 and 2.5 and will sometimes call the first one larger, just because they see more numbers. Be sure to clear this up.

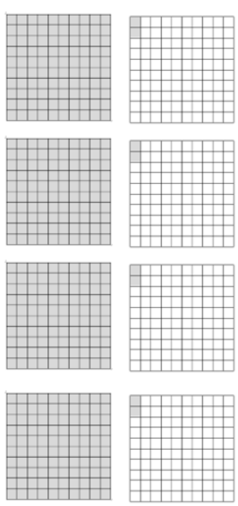


0.6 vs 0.06

This model also shows that 0.6 is equivalent to 0.60. Symbolically we could write 0.60 as $\frac{60}{100}$ and reduce the fraction to $\frac{6}{10}$ which is 0.6

$$\boxed{\frac{60 \div 10}{100 \div 10} = \frac{6}{10} \text{ or } 0.6}$$

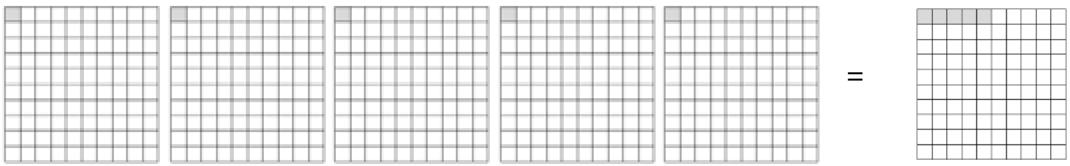
3. We begin multiplying decimals concretely.
Ex: 4 x 1.2



Students can clearly see this is 4.8

Here is where **we teach to use reasoning** to place the decimal in the answer. Students should understand that 4 x 1.2 is close to 4 x 1 so our answer is not 0.48 or 48, but 4.8

Ex. 5 x 0.01 This is 5 groups of $\frac{1}{100}$



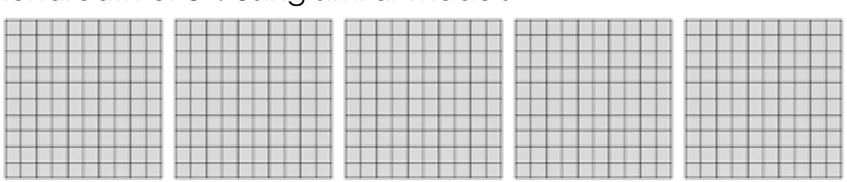
5 x 0.01 = 0.05
Symbolically: When I multiply 5 by a tiny number, I make it smaller.
0.05 is smaller than 5.
The decimal moved *left*

5.

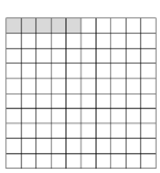
with each "dip" of "moving the decimal
I say "decimal one", "decimal zero one"!
Then I say we fill the egg crate with zeros.

.05.

If we write the factors the other way around, 0.01 x 5, we can think of "one one-hundredth of 5". Using similar models



There are 500 unit squares here in 5 wholes. If I take one one-hundredth of them, that is 5 unit squares



Again, 5 out of 100 or 0.05

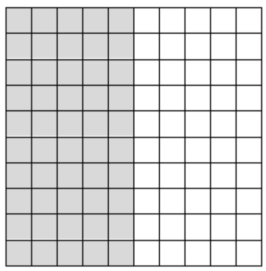
Using area models to multiply two decimal numbers is less straight forward:

Ex: 0.8 x 0.5

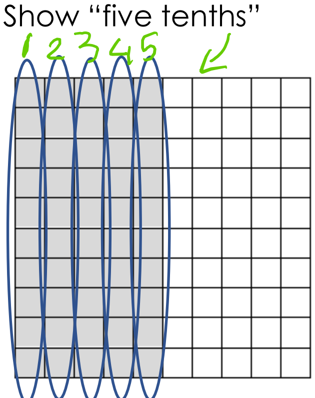
You can find several variations on using area model representation for this, such as paper folding, or overlapping shading. I like to keep relating this to taking "part of something" like fraction multiplication. We are saying 8 tenths of 5 tenths.

Here's how we represent it.

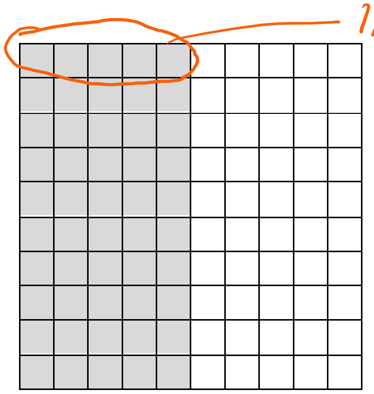
Here are five tenths, or 0.5



Show "five tenths"



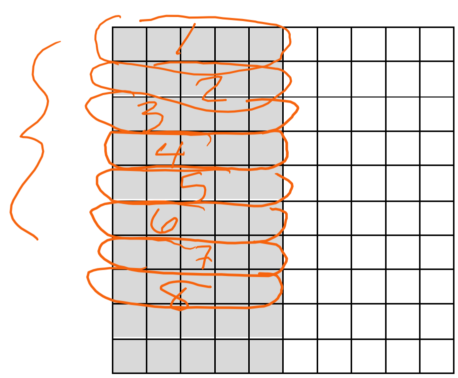
Now I have to take eight tenths of the shaded region. Eight tenths of 5 tenths, written 0.8 x 0.5



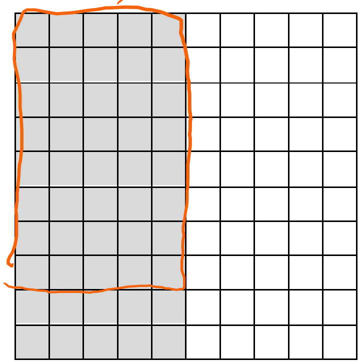
This is one tenth of my shaded region

or $\frac{1}{10}$ of 0.5

I need $\frac{8}{10}$ so



Here is my region 0.8 x 0.5



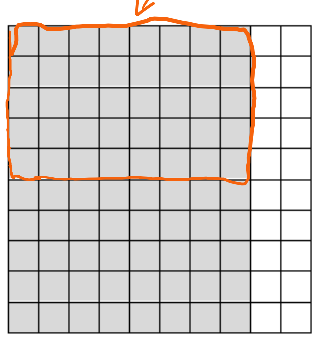
Counting the squares or Seeing the array of 5x8 I see there are 40 unit squares. $\frac{40}{100} = 0.4$

Sense-making: 0.8 of something is a little less than the whole thing, so I expect 0.8 of 0.5 to be a little less than 0.5, so 0.4 makes sense.

By this grade, kids should recognize 0.5 as "half"!

If we reverse the factors, and ask 0.5 x 0.8, we are asking what is "half of 0.8"? So it's logical that it is 0.4

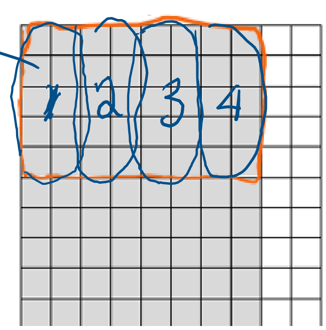
Here is half of 0.8 which is 40/100 or 4/10 or 0.4

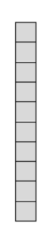


will you need to prove to some students that there are 4 "tenths" in the selected region?

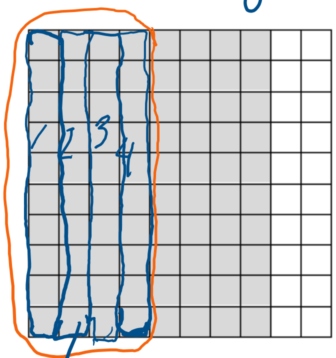
*Student should know from grades 4+5 that 0.40 and 0.4 are equivalent. Be sure they can explain "why"

each of these is ten squares so $\frac{1}{10}$





if kids are struggling to visualize the "tenths" because they're used to seeing "rods" you could select your "half of 0.8" the other way so kids clearly see four "rods".



Four "rods"

It's good to show both arrangements to make more connections and flexible reasoning.

Now practice decimal multiplication, always taking time here and there to draw the models to reinforce the symbolic calculations.

Math Makes Sense texts also show a number line model.

Mental Math and Reasoning:

Ex: 2.5 x 6=
Using reasoning: 2.5 x 6 means we want 2 and a half "sixes". Well 2 sixes is 12, and another half of six is 3, 12 + 3 = 15

Ex: 0.6 x 15=
0.6 x 15 means we want six tenths of 15. One tenth of 15 is 1.5, and we need six of those, so 6 x 1.5 is one and a half sixes. One six is six, and another half is three, so 6 + 3 = 9. (***Notice when we multiplied 15 by a decimal smaller than one, we made is smaller. 9 is smaller than 15*)

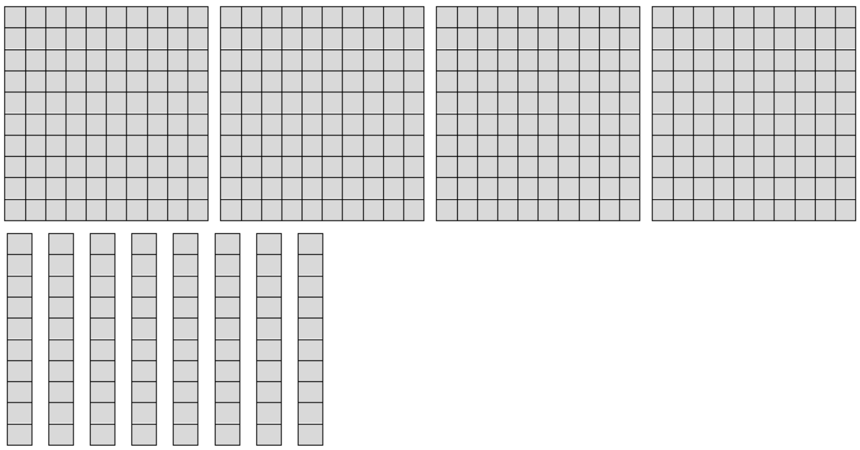
Another way:
0.6 is 0.5 plus 0.1.
So we can take half a number, then a tenth of a number, and add them together
0.6 x 15 =
Take 0.5 of 15 (half of 15) which is 7.5
Take 0.1 of 15 (one tenth of 15) which is 1.5
7.5 + 1.5 is 9
This is very handy for sales tax at 6 % or .06, and also converting miles to km. One km = 0.6 miles. So in the above example, 15 km = 9 miles.

2. Dividing decimals:

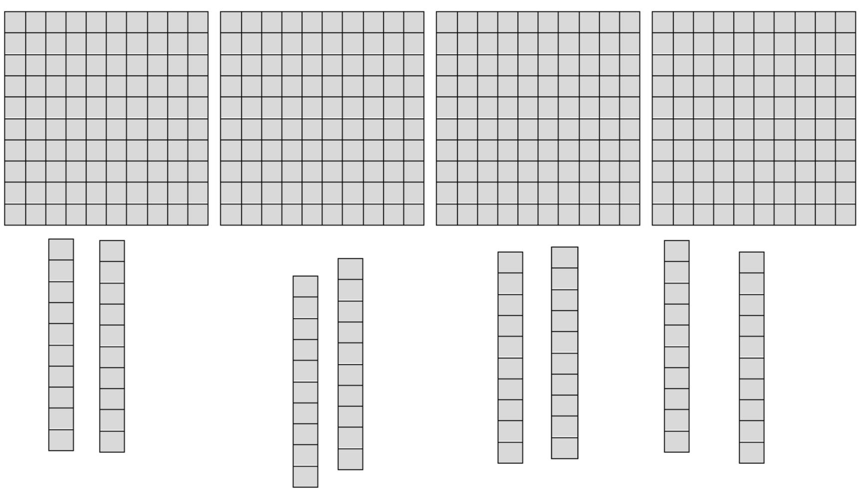
Ex: $4.8 \div 1.2 =$

This is asking "how many "one point twos" are in four point 8?

Lets draw 4.8



You can see we can easily find four groups of 1.2



$4.8 \div 1.2 = 4$

Symbolically: $4.8 \div 1.2$ $\quad 1.2\overline{)4.8}$

we adjust this to

$$\begin{array}{r} 04 \\ 12\overline{)48} \\ 48 \\ \hline 0R \end{array}$$

As fractions: $4\frac{8}{10} \div 1\frac{2}{10}$

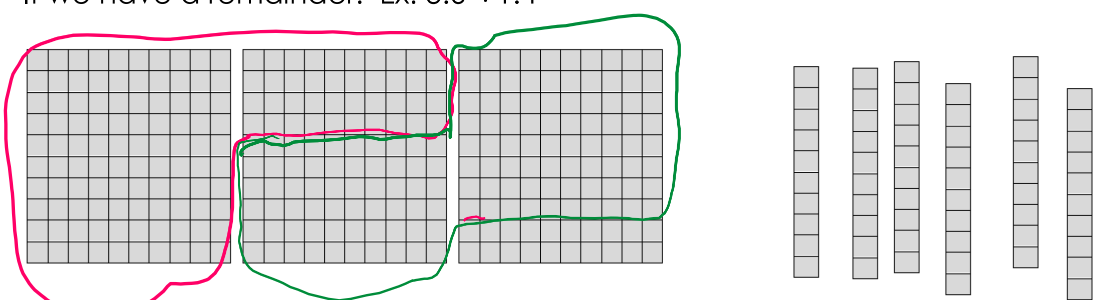
$\frac{48}{10} \div \frac{12}{10}$ (If our denominators are the same, we can just divide the numerators $48 \div 12$ is 4)

or Proceeding with the algorithm: → $\frac{48}{10} \div \frac{12}{10}$

$$\frac{\overset{4}{\cancel{48}}}{\cancel{10}} \times \frac{\cancel{10}}{\cancel{12}} = 4$$

Once or twice it's good to show kids all the ways of conceptualizing this, to make as many connections as possible.

If we have a remainder: Ex: 3.6 ÷1.4 =



2 with 0.8 left over *Note this is NOT 2.8!!

Do we remember what to do with a remainder? We put it over the divisor: $\frac{0.8}{1.4} = \frac{8}{14} = \frac{4}{7}$ so $2\frac{4}{7}$, or we can change this to a decimal.

Algorithm: $3.6 \div 1.4$

$$
\begin{array}{r}
02 \\
14\overline{)36} \\
28 \\
\hline
8
\end{array}
$$

We can move the decimal the same number of spaces in the divisor & dividend.
Why? because
$10 \div 2$ is same as $100 \div 20$
or $1000 \div 200$ etc.

(Note – if we stop here, we put the remainder, 8, over the divisor, 14, so $2\frac{8}{14}$ or $2\frac{4}{7}$. But since the question is given in decimals, we report our answer in decimals)

$$
\begin{array}{r}
02.571 \text{ etc.} \\
14\overline{)36} \\
28 \\
\hline
80 \\
70 \\
\hline
100 \\
98 \\
\hline
20
\end{array}
$$

Using a number line model—lots of example in MMS

3. Moving the decimal point. Often students just want "rules" or "tricks" to remember. This is not teaching conceptually and provides no understanding, plus these "rules" will fail them. For instance, I've had students tell me "when we divide decimals, we move the decimal place *left*". This is only sometimes true!

Kids need to remember:
(I would put these statements on the board and have students agree on what goes in the blanks)

When I **multiply** by a number *greater than 1(a big number)*, my answer gets __________

When I **multiply** by a number *less than 1 (a tiny number)*, my answer gets __________

When I **divide** by a number *greater than 1 (a big number)*, my answer gets __________

When I **divide** by a number *smaller than 1 (a tiny number)*, my answer gets __________

This is not using "rules", it's using understanding and reasoning. All through the previous work , point these relationships out to kids so they can use this reasoning when it comes to multiplying and dividing by factors of 10

Multiply. Describe any patterns you see.

a) 8.36×10	b) 8.36×0.1
8.36×100	8.36×0.01
8.36×1000	8.36×0.001
$8.36 \times 10\,000$	8.36×0.0001

From MMS 7

Divide. Describe any patterns you see.

a) $124.5 \div 10$	b) $124.5 \div 0.1$
$124.5 \div 100$	$124.5 \div 0.01$
$124.5 \div 1000$	$124.5 \div 0.001$
$124.5 \div 10\,000$	$124.5 \div 0.0001$

I would put up mixed practice

6.2 x 100 =
45.98 x 0.001 =
452 ÷1 000 =
934.522 ÷ 0.01 =

etc