Capacity Building Series

SECRETARIAT
SPECIAL EDITION #13

Communication in the Mathematics Classroom

Gallery Walk, Math Congress and Bansho

In Margo's Grade 3 class, small groups of students create different solutions to a lesson problem and then present their solutions to their classmates. Although the first group's solution includes colourful pictures and a lengthy description, it doesn't explain the strategy they've used. The second group's solution is difficult to follow. As group after group shares, attention begins to fade. As the sixth group presents, Margo sees only a few students listening. As the students go out for recess, Margo wonders, "What are students really learning by sharing? How can I help my students become more effective mathematical communicators?"

What do you do to develop your students' mathematical communication? Do you choose mathematics tasks and problems that evoke significant mathematics and motivate students to discuss their mathematical thinking? Maybe you provide time for students to discuss and hear the mathematical ideas of other students. Perhaps you record the mathematical details generated from whole-class discussion on the chalkboard. Are you wondering if a prompt like, "Use words, numbers and pictures" oversimplifies the processes involved in communicating, precisely and succinctly, the intricate details of mathematical thinking?"

Developing effective mathematical communication

The development of students' mathematical communication shifts in precision and sophistication throughout the primary, junior and intermediate grades, yet the underlying characteristics remain applicable across all grades. During whole-class discussion, teachers can use these characteristics as a guide both for interpreting and assessing students' presentations of their mathematical thinking and for determining discussion points.

Why is mathematical communication important?

Mathematical communication is an essential process for learning mathematics because through communication, students reflect upon, clarify and expand their ideas and understanding of mathematical relationships and mathematical arguments.

(Ontario Ministry of Education, 2005)

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Categories of Mathematical Communication ...

- *expression and organization of ideas and mathematical thinking* (e.g., clarity of expression, logical organization), using oral, visual, and written forms (e.g., pictorial, graphic, dynamic, numeric, algebraic forms; concrete materials)
- *communication for different audiences* (e.g., peers, teachers) and purposes (e.g., to present data, justify a solution, express a mathematical argument in oral, visual, and written forms)
- *use of conventions, vocabulary and terminology of the discipline* (e.g., terms, symbols) in oral, visual, and written forms

(Ontario Ministry of Education, 2005, p. 23)

Talk about mathematics doesn't come naturally ...

"Because mathematics is so often conveyed in symbols, oral and written, communication about mathematical ideas is not always recognized as an important part of mathematics education. Students do not necessarily talk about mathematics naturally; teachers need to help them learn how to do so."

(Cobb, Wood, & Yackel, 1994)

The characteristics, listed below, are relevant across grades:

- ***precision*** about problem details, relevant choice of method or strategy to solve the problem, accurate calculations
- ***assumptions and generalizations*** that show how the details of the mathematical task/problem are addressed in the solution
- ***clarity*** in terms of logical organization for the reader's ease of comprehension, requiring little or no reader inference
- a ***cohesive argument*** that consists of an interplay of explanations, diagrams, graphs, tables and mathematical examples
- ***elaborations*** that explain and justify mathematical ideas and strategies with sufficient and significant mathematical detail
- appropriate and accurate use of ***mathematical terminology, symbolic notation and standard forms*** for labelling graphs and diagrams.

Organizing students to think, talk and write

Through listening, talking and writing about mathematics, students are prompted to organize, re-organize and consolidate their mathematical thinking and understanding, as well as analyze, evaluate and build on the mathematical thinking and strategies of others. The use of mathematical language helps students gain insights into their own thinking and develop and express their mathematical ideas and strategies, precisely and coherently, to themselves and to others.

It is during whole-class discussion that students explain and justify their ideas and strategies as well as challenge and ask for clarification from their classmates. However, coordinating whole-class discussion in a timely manner, in order to develop students' mathematical understanding, is demanding mathematical work (Ball, Thames, & Phelps, 2009). Such coordination requires the teacher to carry out several mathematically-based pedagogical moves simultaneously :

- coaching students on how to participate in mathematical discussions (e.g., questioning, explaining, probing one another's mathematical reasoning)
- developing from and expanding on students' mathematical solutions to make explicit mathematical concepts and strategies related to the lesson's goal
- creating mathematical visual records of the class discussion for all students to see
- using mathematical notation to record students' mathematical thinking – a way for primary students to learn that "writing is thinking, written down"

When teacher talk dominates whole-class discussion, students tend to rely on teachers to be the expert, rather than learning that they can work out their own solutions and learn from other students. Gallery Walk, Math Congress and Bansho provide students with organized and facilitated time to talk about and listen actively to one another's mathematical thinking, justify their thinking to others and reflect on what they are learning. In fact, this organized and safe discussion forum encourages students to share and challenge ideas. Importantly, students are reassured that their voices, ideas and experiences are valued and contribute directly to the whole class learning.

During Gallery Walk, Math Congress and Bansho, it becomes evident to the students and teacher that mathematical communication is not about "answering the question using words, numbers, pictures, and symbols." Instead, they realize that these forms of communication are selected and applied in order to create a precise mathematical argument, where labelled diagrams and/or numeric expressions and equations are viewed as being more precise, concise and persuasive forms than descriptive narratives. These discussion processes provoke students to use higher-order thinking skills, such as analysis, evaluation and synthesis, in order to improve their conceptual understanding, use of mathematical strategies and mathematical communication.

Updating the three-part problem-solving lesson

In the pages that follow, the three approaches for co-ordinating student discussion and analysis of solutions – Gallery Walk, Math Congress and Bansho – are illustrated within the framework of the three-part problem-solving lesson. The entire lesson should take between 45 and 60 minutes.

1. ***Before – Getting started*** (5 to 10 minutes). Revisiting mathematical ideas and strategies from a previous lesson that relates to the learning goal of the lesson
2. ***During – Teaching/learning*** (15 to 20 minutes). Solving the lesson problem in pairs, small groups or individually
3. ***After***

   (a) ***Consolidation*** (20 to 25 minutes). Co-ordination of whole-class discussion and analysis of student solutions

   (b) ***Highlights/summary*** (5 minutes). Recounting key mathematical ideas and strategies related to the learning goal of the lesson

   (c) ***Practice*** (5 to 10 minutes). Solving a problem that is similar to the lesson problem in order to practise applying new ideas and strategies

Gallery Walk

Gallery Walk is an interactive discussion technique that gets students out of their chairs and into a mode of focused and active engagement with other students' mathematical ideas (Fosnot & Dolk, 2002). The purpose of the Gallery Walk is to have students and the teacher mathematically engage with a range of solutions through analysis and response. It is often carried out after students have generated solutions to a mathematics lesson problem. Solutions could be recorded on computers, pieces of paper on tables or posted chart paper. A Gallery Walk is often scheduled for about 10 to 20 minutes depending on the instructional purpose and depth of mathematical analysis expected.

For students, Gallery Walk is a chance to read different solutions and provide oral and written feedback to improve the clarity and precision of a solution. On the other hand, for teachers, it is a chance to determine the range of mathematics evident in the different solutions and to hear students' responses to their classmate's mathematical thinking. Such assessment for learning data help the teacher to determine points of emphasis, elaboration and clarification for the ensuing whole class discussion (Fosnot & Dolk, 2002).

Although there are different variations of a Gallery Walk, a common approach is outlined below:

(a) ***Small-group problem solving*** – Students, in small groups, develop one solution to the lesson problem on chart paper.

(b) ***Small-group discussion*** – Small groups take turns reading and analyzing one another's solutions and recording comments, questions and/or suggestions for improvement, using stick-on notes (for later sorting) or writing directly on the chart paper. After three to five minutes, the groups rotate to the next solution. Rotation continues until all solutions are analyzed and responded to by all groups. As comments accumulate for each solution, the groups also review what previous groups have written and add only new comments, questions and/or suggestions for improvement.

(c) ***Teacher observation*** – As students are discussing their classmates' solutions, the teacher circulates around the classroom, gauging student understanding and noting students' use of mathematics vocabulary and symbolic notation as well as their mis-matched conceptions.

Students develop their mathematical communication in different ways ...

For examples, see the following LNS WEBCASTS:

- Making Mathematics Accessible for All Students
- Coaching for Student Success in Mathematics
- Learning Mathematics Within Contexts
- Investigating High Yield Strategies for Improving Mathematics Instruction and Students' Learning
- Differentiating Mathematics Instruction
- Understanding Geometric Figures Through Drawing and Paper Folding

www.curriculum.org/csc/learning.shtml

(d) ***Whole-class discussion*** – When the groups return to their own solution, they synthesize the comments, questions and suggestions for improvement into an oral report that will be presented to the whole class. Small-group oral reports provide specific details that the teacher can use to highlight and summarize key mathematical ideas and strategies related to the lesson learning goal as well as include discussion about mathematical misconceptions and errors. Also, the group can apply their classmate's responses to revise their solution.

Math Congress

Math Congress is a mathematics instructional strategy developed by Fosnot and Dolk (2002). Preparation for and participation in a Math Congress occurs over two lesson periods. The purpose of the congress is to support the development of mathematicians in the classroom learning community, rather than fixing mistakes in the children's work or getting agreement on answers. A congress enables the teacher to focus the students on reasoning about a few big mathematical ideas derived from the mathematical thinking present in the students' solutions. Therefore, a Math Congress is not about showing every solution, as there is not enough time, nor is every student at the same place where the strategy will make sense to them. Instead, it focuses whole-class discussion on two or three, strategically selected, student solutions in order to develop every student's mathematical learning. To explore this strategy, try solving this problem yourself in two different ways. This problem is designed by Jacob and Fosnot (2007).

See a Math Congress in action ...

- High-Yield Strategies for Improving Mathematics Instruction and Student Learning

  http://www.curriculum.org/secretariat/february26.shtml

Cat Food Problem: Kittens have to eat a special kind of cat food. There are two stores that sell this kind of cat food. The cans are the same size and same brand. Which one is the better deal? Show your work.

Bob	12 cans for $15.00
Maria	20 cans for $23.00

How does your solution compare to these solutions? How are they similar? Different?

Solutions to the Cat Food Problem

Using Division

Bob's store	
12÷2=6	→ $15÷2 = $7.50
6÷2=3	→ $7.5÷2 = $3.75
3÷3=1	→ $3÷3 = $1 and $0.75÷3 = $0.25
Maria's store	
20÷2=10	→ $23÷2 = $11.50
10÷2=5	→ $11.50÷2 = $5.75
5÷5=1	→ $5÷5 = $1 and $0.75÷5 = $0.15

Using Division

Bob's store
$15.00÷12 cans
= $1.25 per can

Maria's store
$23.00÷20 cans
= $1.15 per can

Common Whole

Bob's store	
12 cans	$15.00
X5	x 5
60 cans	$75.00
Maria's store	
20 cans	$23.00
X 3	x 3
60 cans	$69.00

Ratio Table

Cans	Bob	Maria
12	$15.00	?
6	$7.50	?
3	$3.75	?
1	$1.25	$1.15
20	?	$23.00
60	$75.00	$69.00
10	?	$11.50
5	?	$5.75

For the *Cat Food Problem*, two big ideas could be the focus of the Math Congress:

1. unit price results from division – the total price divided by the number of cans
2. equivalence – if you multiply (or divide) the price and the number of items you get for that price by the same number, you obtain an equivalent pricing for a different number of items (e.g., common whole of 60 cans)

Since these two strategies correspond to the operations used in a ratio table (i.e., multiplication, division), the teacher can introduce a ratio table by organizing the different ratios that students used to compare the cost of the cat food. Also, students could be challenged to consider how to use the ratio table in order to determine numbers that are not already listed on the table (Fosnot & Dolk, 2002; Fosnot, 2007). A common process for a Math Congress (Fosnot & Dolk, 2002; Fosnot, 2007) is as follows:

Preparing for the Math Congress – Pairs of students make chart paper size posters of their solutions. The posters should be concise, clear presentations of the important ideas and strategies that students want to communicate to their classmates. During this time, the teacher notes students' use of mathematical strategies and ideas in preparation for the Math Congress. The teacher imagines how the whole-class discussion could ensue, using these questions:

- What mathematical ideas and strategies in the posters should be discussed?
- How do these ideas and strategies relate to student learning of the lesson learning goal, as well as build on previous mathematical discussions?
- Which ideas and strategies can be generalized? How might mathematical generalization be provoked?
- What is a possible sequence for the discussion of the posters, so that it serves as a scaffold for learning?

Mini-Congress – To prepare for the Math Congress, students share their work with one another, check answers and strategies, and ask questions to provoke clarification and/or elaboration. During these small-group discussions, all students share their strategies, listen to the ideas of others, question what they do not understand and defend their thinking. One student in each group facilitates this discussion, making sure that all strategies are shared and that everyone in the group has asked and/or responded to a question.

Facilitating the Math Congress – The whole class gathers together to discuss solutions presented by two or three pairs of students. Students defend and support their mathematical thinking as the teacher guides the whole-class discussion toward important mathematical ideas and strategies. In order to prompt students to reason and make mathematical generalizations, these sorts of questions are posed: "How is this strategy similar to and different from the first solution present? Will this strategy always work? How do you know? When will it not work? Why not?"

Teachers need to build on the personal and collective sense-making of students ...

"The role of the teacher during whole-class discussion is to develop and the build on the personal and collective sense-making of students rather than to simply sanction particular approaches as being correct or demonstrate procedures for solving predictable tasks."

(Stein, Engle, Smith, & Hughes, 2008, p. 315)

Bansho (Board Writing)

Bansho, in Japanese, literally means board writing. According to Yoshida (2002) and Fernandez and Yoshida (2004), the purpose of Bansho is to organize and record mathematical thinking derived from and collectively produced by students on a large-size chalkboard or dry erase board. Such board writing includes the use of mathematical expressions, figures and diagrams of students' solutions and strategies to a lesson problem. Because this written record enables simultaneous comparison of multiple-solution methods, there is the potential for students to construct new mathematical ideas and deepen their mathematical understanding. Because the chalkboard is a written record of the entire lesson, the students and teacher have a whole view of the class' mathematical discussion throughout the lesson. Also, by modelling effective organization, Bansho fosters student note-taking skills. Importantly, Japanese teachers aim to keep all that is written throughout the lesson on the chalkboard without erasing (Yoshida, 2002).

Japanese Bansho has been interpreted and adapted within the context of teaching and learning mathematics in Ontario by Kathryn Kubota-Zarivnij. She outlines the specific aspects of (Ontario) Bansho in the context of the three-part problem-solving lesson.

Teachers as the consciousness of the collective ...

"The teacher is responsible for prompting differential attention, selecting among the options for action and interpretation that arise in the collective … teaching cannot be about zeroing in on predetermined conclusion. It can't be about the replication and perpetuation of the existing possible. Rather, teaching seems to be more about expanding the space of the possible and creating conditions of the emergence of the as-yet-unimagined."

(Davis, 2005, p. 87)

How should the board be organized?

A large, cleared, public writing space (e.g., chalkboard, dry-erase-board or large piece of mural or butcher paper, but not chart paper stands) is needed for the recording of the mathematical details generated throughout the lesson.

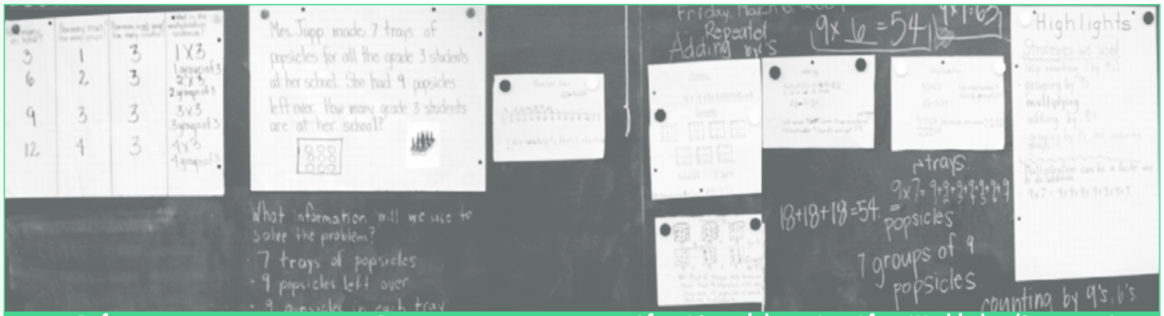
1. ***Before – Getting started.*** The teacher records the discussion prompt or activating problem and the students' mathematical responses, highlighting details that link to the lesson's mathematics learning goal. About 1/8 of board space is needed.
2. ***During – Teaching/learning.*** The teacher records the lesson problem and a list of the information that the class identifies to use when making a plan to solve the problem. Student solutions need to be readable by others. Provide at least "11x17" paper in landscape orientation, using markers. About 1/8 of board space is needed.
3. ***After***

   (a) ***Consolidation.*** Two to four different solutions are posted on the board as a visual for students to use to explain their solutions and for other students and the teacher to offer questions and comments. The teacher records these mathematical elaborations or mathematical annotations on and around the solutions, so that the mathematical thinking behind the solution is explicit to everyone. About 1/2 of board space is needed.

   (b) ***Highlights/summary.*** Key mathematical concepts, algorithms and/or strategies related to the lesson learning goal are summarized and recorded in a list so that the learning from the lesson is explicit to all students. About 1/8 of board space is needed.

   (c) ***Practice.*** One of the practice problems and two student solutions to it are recorded on the board. About 1/8 of board space is needed.

GRADE 3 BANSHO FROM A MULTIPLICATION LESSON



Before | During | After (Consolidation) | After (Highlights/Summary)

Which student solutions should be selected?

As students work on the problem, the teacher chooses two to four different solutions that can be knitted together to develop an understanding of the lesson learning goal during upcoming whole-class discussion. In Japanese classrooms, student solutions are selected according to three criteria: accuracy, efficiency and generalization. In Ontario classrooms, broader selection criteria are used in order to emphasize the notion of mathematical elaborations; that is, how one solution builds on and leads to the mathematics inherent in another solution. Such criteria could include type of mathematical strategy (e.g., for addition, counting on, joining numbers to make a new number, making fives and tens, doubles 1 more than or 1 less than); relationship between representations of a concept and alternative and standard algorithms; types of numeric expression.

Which solutions should be shared first?

Rather than having students randomly volunteer to share their solutions, the solutions need to be strategically sequenced to scaffold learning. Student solutions are not organized by levels of achievement. Instead, generally, conceptually-based solutions are shared first, followed by those solutions that include more efficient strategies or algorithms, followed by solutions that focus on generalizations. To decide which solution should be shared first, second and third, a teacher asks her/himself these questions:

How should classroom discourse be organized?

- What mathematics (i.e., concept, algorithm, strategy, model of representation) are the students using in their solution? How does this mathematics in the solution relate to the mathematics lesson learning goal?
- Which solutions are conceptually based? Which solutions have an efficient method or algorithm? Which solutions include or have the potential for a mathematical generalization?
- How are the solutions related to one another, mathematically? How are the solutions related to the mathematics learning goal of the lesson?

Students (authors of the selected solutions) present a summary of the strategy they used to solve the problem. Their classmates ask questions and make comments about the solution details for the purpose of making explicit the mathematical detail in the solutions. The teacher records significant details of this discourse concisely on and around the solutions on the board.

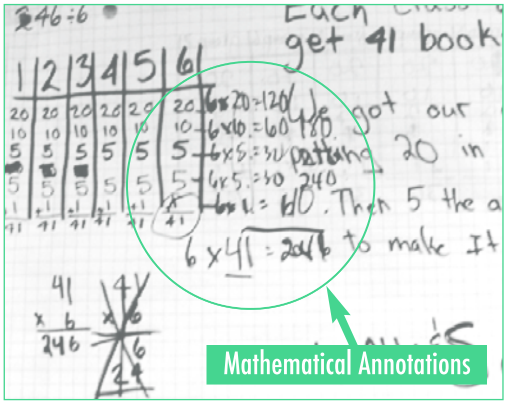
What mathematical annotations are used?

The teacher records mathematical annotations on and around the student work for several reasons:

(a) to elaborate on and make explicit mathematical thinking that wasn't evident in students' written communication, using labelled diagrams, tables, concise explanatory details

(b) to formalize students' ideas using mathematical terms and notation (symbols, numerals)

(c) to highlight mathematical connections between students' solutions, so that students can see how solutions enfold from and into one another.



For example, the mathematical annotations illustrate how the data in the chart can be recorded as numerical equations; that is, a sum of multiplication sentences.

Some tips on getting started

Organizing the classroom learning environment

It takes ongoing, intentional work to create and reinforce a classroom culture in which all students feel comfortable exposing their thinking in front of their peers when they question, react to and elaborate on the statements of their classmates and the teacher. Some suggestions for furthering this work follow:

- Have students sit in flexible groupings face to face in pairs or small groups, with sufficient space for small-group writing and the use of concrete materials.
- Clear the board (i.e., chalkboard, write-erase board or post substantial length [2m] of mural or butcher paper) so there is ample space for posting student work and/or recording mathematical details throughout the entire three-part problem-solving lesson.
- Use co-operative learning strategies, like Turn and Talk, Think-Pair-Share, Round Table, Think-Talk-Write and Place Mat, to organize student interaction/discussion and to provide wait time for students to formulate a response.
- Develop a culture of listenership where students (and the teacher) listen to every student in a nonjudgmental, inquisitive and attentive way.

Why oral and written communication is important in the mathematics classroom ...

"Oral Communication includes talking, listening, questioning, explaining, defining, discussing, describing, justifying, and defending. When students participate in these actions in an active, focused, and purposeful way, they are furthering their understanding of mathematics."

(Ontario Ministry of Education, 2006, p. 66)

"Written Communication enables students to think about and articulate what they know. Mathematical writing also provides evidence of students' mathematical understanding. Before beginning any writing task, students need experiences in expressing their ideas orally, as well as listen to the ideas of others. The quality of a written product is significantly improved by the opportunity to participate in a class dialogue before writing."

(Ontario Ministry of Education, 2006)

Preparing yourself mathematically

- Solve the mathematics lesson problem in at least two different ways, in order to get a mathematical sense of the concepts, skills, and strategies that the problem could evoke.
- Use the Ontario mathematics curriculum expectations (Ontario Ministry of Education, 2005) and the *Guides to Effective Instruction in Mathematics* to develop a list of the mathematics terms, models of representation and symbolic notation related to the lesson (and unit of study).
- Ask yourself these questions as you look at and listen to students' mathematical thinking:
  - What mathematics is evident in students' communication (oral, written, modelled)?
  - What mathematical language should we use to articulate the mathematics we see and hear from students? (e.g., mathematical actions, concepts, strategies)
  - What mathematical connections can be discerned between students' different solutions?

Coordinating Student Discussion and Analysis of Solutions

- Listen actively to students' ideas in order to notice common understandings that can be established, as well as to determine when to step in and out of the discussion, when to press for understanding, when to resolve competing student claims and when to address misunderstandings or confusion.
- Ask clarifying and extending questions that prompt clarification, that gather and generate different ideas and approaches and that challenge the validity of ideas discussed. Sample open-ended questions include: "How do you know?" "Why?" "What if ...?" "What do you notice?" Expanding questions include: "What's another solution, strategy or explanation? And another? And another?" These sorts of questions prompt all students to think about and make connections between their mathematical thinking and the class mathematical discussion.
- Prompt revoicing, such as student and/or teacher retelling, rephrasing and/or expanding on students' mathematical ideas. Revoicing highlights ideas that have come directly from students and it provides a space to negotiate and delve deeply into the implicit meanings inherent in students' mathematical ideas and questions.

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